The set rational and irrational numbers | Answers

Definition 1: A rational number is a number that can be written in the form of a fraction $\frac{a}{b}$ such that a and be are integers and $b \neq 0$.

Definition 2: A natural number $\boldsymbol{n}$ is called a **perfect square** if there is an integer $\boldsymbol{a}$ such that $\boldsymbol{n} = \boldsymbol{a}^2$.

Property of perfect squares:

A natural number $\boldsymbol{n}$ greater than 1 is a perfect square if and only if in the prime factorization of the number $\boldsymbol{n}$ every prime number occurs an **even** number of times.

Example 1. $2 \times 2 \times 7 \times 7 \times 7 \times 7 \times 13 \times 13$ is a perfect square.

Example 2. $2 \times 2 \times 7 \times 7 \times 7 \times 13 \times 13$ is not a perfect square.

The property of perfect squares may be used to prove that some numbers are not rational.

Task 1. Prove that the following numbers are not rational. The example (a) is done for you.

(a) $\sqrt{40}$ (b) $\sqrt{18}$ (c) $\sqrt{2}$ (d) $\sqrt{7}$

(b) **PROOF by contradiction**

We have to prove that $\sqrt{18}$ is not a rational number,
but **suppose that $\sqrt{18}$ is a rational number**.

This means that there are two integers a and b such that $\sqrt{18} = \frac{a}{b}$ and so:

$$\sqrt{18} = \frac{a}{b}$$

$$\left(\sqrt{18}\right)^2 = \left(\frac{a}{b}\right)^2$$

$$18 = \frac{a^2}{b^2}$$

$$18b^2 = a^2$$

$$2 \times 3 \times 3 \times b^2 = a^2$$

By **property of perfect squares,** in prime factorization of the number $\boldsymbol{b}^2$ the prime number **2** occurs an **even** number of times. Therefore in the prime factorization of the number $\mathbf{2 \times 3 \times 3} \times \boldsymbol{b}^2$ the prime number **2** occurs an **odd** number of times, but this means that the perfect square $\boldsymbol{a}^2$ does not satisfy the property of perfect squares and this is **impossible**. This contradicts our initial supposition, so we are forced to conclude that $\sqrt{18}$ is not a rational number.

END OF THE PROOF

(c) PROOF by contradiction

We have to prove that $\sqrt{2}$ is not a rational number,
but **suppose that $\sqrt{2}$ is a rational number**.

This means that there are two integers a and b such that $\sqrt{2}=\frac{a}{b}$ and so:

$$\sqrt{2}=\frac{a}{b}$$

$$\left(\sqrt{2}\right)^2=\left(\frac{a}{b}\right)^2$$

$$2=\frac{a^2}{b^2}$$

$$2b^2=a^2$$

$$2\times b^2=a^2$$

By **property of perfect squares,** in prime factorization of the number $\boldsymbol{b^2}$ the prime number **2** occurs an **even** number of times. Therefore in the prime factorization of the number $\mathbf{2}\times\boldsymbol{b^2}$ the prime number **2** occurs an **odd** number of times, but this means that the perfect square $\boldsymbol{a^2}$ does not satisfy the property of perfect squares and this is **impossible**.
This contradicts our initial supposition, so we are forced to conclude that $\sqrt{2}$ is not a rational number.

END OF THE PROOF

(d) PROOF by contradiction

We have to prove that $\sqrt{7}$ is not a rational number,
but **suppose that $\sqrt{7}$ is a rational number**.

This means that there are two integers a and b such that $\sqrt{7}=\frac{a}{b}$ and so:

$$\sqrt{7}=\frac{a}{b}$$

$$\left(\sqrt{7}\right)^2=\left(\frac{a}{b}\right)^2$$

$$7=\frac{a^2}{b^2}$$

$$7b^2=a^2$$

$$7\times b^2=a^2$$

By **property of perfect squares,** in prime factorization of the number $\boldsymbol{b^2}$ the prime number **7** occurs an **even** number of times. Therefore in the prime factorization of the number $\mathbf{7}\times\boldsymbol{b^2}$ the prime number **7** occurs an **odd** number of times, but this means that the perfect square $\boldsymbol{a^2}$ does not satisfy the property of perfect squares and this is **impossible**.

This contradicts our initial supposition, so we are forced to conclude that $\sqrt{7}$ is not a rational number.

END OF THE PROOF

Task 2. Fill in each of the following tables.

Use "Y" for yes if the row name applies to the number or "N" for no if it does not.

Number	$-\sqrt{16}$	$\sqrt{7}$	3.14	-1	0	$\frac{\sqrt{2}}{2}$	$\frac{5}{0}$	$4.\overline{25}$	11	$\frac{2}{-3}$	$2\frac{1}{5}$	6
Undefined	N	N	N	N	N	N	**Y**	N	N	N	N	N
Positive natural	N	N	N	N	N	N	-	N	**Y**	N	N	**Y**
Integer	**Y**	N	N	**Y**	**Y**	N	-	N	**Y**	N	N	**Y**
Rational	**Y**	N	**Y**	**Y**	**Y**	N	-	**Y**	**Y**	**Y**	**Y**	**Y**
Irrational	N	**Y**	N	N	N	**Y**	-	N	N	N	N	N
Prime	N	N	N	N	N	N	-	N	**Y**	N	N	N
Composite	N	N	N	N	N	N	-	N	N	N	N	**Y**

Task 3. Find two irrational numbers a, b such that their sum $a + b$ is a rational number, but their difference $a - b$ is an irrational number. One example is done for you. Think by analogy and find another example.

	a	b	$a + b$	$a - b$
Example 1	π	$-\pi$	0	2π
Example 2	$\sqrt{5}$	$3 - \sqrt{5}$	3	$2\sqrt{5} - 3$

Task 4. Find two irrational numbers a, b such that their product $a \times b$ is a rational number and their quotient $a \div b$ is a rational number. One example is done for you. Think by analogy and find another example.

	a	b	$a \times b$	$a \div b$
Example 1	$\sqrt{2}$	$\sqrt{18}$	6	3
Example 2	$\sqrt{32}$	$\sqrt{2}$	8	4